

Pulsating strings in warped $AdS_6 \times S^4$ geometry

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Abstract

In this paper we consider pulsating strings in warped $AdS_6 \times S^4$ background, which is a vacuum solution of massive type **IIA** superstring. The case of rotating strings in this background was considered in hep-th/0402202 and it was found that the results significantly differs from those considered in $AdS_5 \times S^5$. Motivated by this results we study pulsating strings in the warped spherical part of the type **IIA** geometry and compare the results with those obtained in hep-th/0209047, hep-th/0310188 and hep-th/0404012. We conclude with comments on our solutions and the obtained corrections to the energy, expanded to the leading order in lambda.

1 Introduction

In the last several years the main efforts towards establishing string/gauge theory duality were focused on $AdS_5 \times S^5$. This background has some particularly interesting features: it is supersymmetric with maximal amount of supersymmetry, it is dual to $N = 4$ SYM theory, the correspondence between string states and operators on gauge theory side is particularly well defined, etc. Shortly after the Maldacena conjecture was established, the supergravity

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approximation was intensively studied, but it was realized that the correspondence requires investigations beyond that limit. The study in this direction was developed mainly in two ways - pp-wave limit of the background geometry (see for example [1]) and semiclassical strings in these geometries [2]. The main idea developed in [2] was based on the following. String theory in the most string backgrounds is highly non-linear and it is impossible by now to directly attack the problem. One possibility is to consider semiclassical strings in these backgrounds. In order AdS/CFT correspondence to be valid, one must consider strings carrying high energy and large momenta. The simplest way to investigate string/gauge theory correspondence is to consider an ansatz corresponding to particular string configurations which makes the problem tractable. On other side $N = 4$ SYM was also intensively studied, for instance by making use of Bethe ansatz technique [9, 7, 11]. Comparison of the results on both sides gave good agreement.

The AdS/CFT correspondence in less supersymmetric backgrounds was considerably less studied [5, 28, 24, 25, 27, 4]. These backgrounds however are actually more suitable for description of hadron physics [26] and therefore are of particular interest. The main difficulty in working with these backgrounds is that it is not completely clear which string states to which gauge theory operators correspond. Nevertheless, it is still interesting to study the string theory side and to collect results for future applications.

In this paper we focus on type **IIA** $AdS_6 \times S^4$ background which contains warp factor. The method of rotating strings for $AdS_5 \times S^5$ background developed in [3]² was successfully applied to this case in [28]³. It was found that the anomalous dimensions for the operators in the dual D=5 $N = 2$ gauge theory qualitatively differ from those obtained in the case of strings in $AdS_5 \times S^5$. For the short strings one has the same behavior in both cases

$$E = \sqrt{2J} \left(1 + \frac{1}{8}J + \frac{3}{128}J^2 + \frac{1}{1024}J^3 + \dots \right). \quad (1.1)$$

For long strings in the former case E and J both diverge logarithmically as $\log\epsilon$, while in the latter case $E - J$ approaches a constant

$$E - J = \frac{2}{\pi} - \frac{8}{\pi}e^{-\pi J-2} + \dots \quad (1.2)$$

All these show that for short strings, located near the north pole the effect of warp factor is negligible and does not affect the behavior of E and J .

²For recent review see [30].

³See also [24].

In the case of long strings, due to the warp factor, one has repulsion from the equatorial boundary which restricts the rotation to the S^3 part. This is the reason why D=5 corresponding to $AdS_6 \times S^4$ Yang-Mills theory is significantly different from D=4 YM corresponding to $AdS_5 \times S^5$.

There is however one more important class of string solutions that are important and it corresponds to the cases considered in [10, 13, 11, 29], namely pulsating strings. In this paper we consider pulsating strings in the warped $AdS_6 \times S^4$. In the first Section we review the method of pulsating strings in $AdS_5 \times S^5$ background. In Section 3 we consider pulsating strings in the warped S^4 sphere and find the expression for the energy E and its quantum corrections. In the Conclusions we briefly outline the results and comment on some open questions.

2 Review of pulsating strings in $AdS_5 \times S^5$

In this section we give a brief review of the pulsating string solutions obtained first by Minahan [10] and generalized later in [11, 13]. For later use we will concentrate only on the case of pulsating string on the S^5 part of $AdS_5 \times S^5$, i.e. we consider a circular string expanding and contracting on S^5 . We start with the metric of S^5 and the relevant part of AdS_5

$$ds^2 = R^2 (\cos^2 \theta d\Omega_3^2 + d\theta^2 + \sin^2 \theta d\psi^2 + d\rho^2 - \cosh^2 dt^2), \quad (2.1)$$

where $R^2 = 2\pi\alpha'\sqrt{\lambda}$. In the simplest case, we identify the target space time with the worldsheet one, $t = \tau$, and use the ansatz $\psi = m\sigma$, i.e. the string is stretched along ψ direction, $\theta = \theta(\tau)$. We will keep the dependence of ρ on τ for a while, $\rho = \rho(\tau)$. The reduced Nambu-Goto action in this case then is

$$S = m\sqrt{\lambda} \int dt \sin \theta \sqrt{\cosh^2 \rho - \dot{\theta}^2}. \quad (2.2)$$

Since we are interesting in calculating the energy, it is useful to pass to Hamiltonian formulation. For this purpose we find first the canonical momenta and the Hamiltonian in the form⁴

$$H = \cosh \rho \sqrt{\Pi_\rho^2 + \Pi_\theta^2 + m^2 \lambda \sin^2 \theta}. \quad (2.3)$$

⁴For more details see [10].

Fixing the string to be at the origin of AdS_5 space ($\rho = 0$), one can observe that the squared Hamiltonian have a form very similar to a point particle. The last term in (2.3) can be considered as a perturbation to the free Hamiltonian, so first we find the wave function for the free theory in the above geometry

$$\begin{aligned} \frac{\cosh\rho}{\sinh^3\rho} \frac{d}{d\rho} \cosh\rho \sinh^3\rho \frac{d}{d\rho} \Psi(\rho, \theta) - \frac{\cosh^2\rho}{\sin\theta \cos^3\theta} \frac{d}{d\theta} \sin\theta \cos^3\theta \frac{d}{d\theta} \Psi(\rho, \theta) \\ = E^2 \Psi(\rho, \theta). \end{aligned} \quad (2.4)$$

The solution to the above equation is

$$\Psi_{2n}(\rho, \theta) = (\cosh\rho)^{-2n-4} P_{2n}(\cos\theta) \quad (2.5)$$

where $P_{2n}(\cos\theta)$ are spherical harmonics on S^5 and the energy is given by

$$E_{2n} = \Delta = 2n + 4. \quad (2.6)$$

Since we consider highly excited states, one should take large n , so one can approximate the spherical harmonics by simple trigonometric functions

$$P_{2n}(\cos\theta) \approx \sqrt{\frac{4}{\pi}} \cos(2n\theta). \quad (2.7)$$

The correction to the energy can be obtained by making use of perturbation theory, which to first order gives

$$\delta E^2 = \int_0^{\pi/2} d\theta \Psi_{2n}^*(0, \theta) m^2 \lambda \sin^2\theta \Psi_{2n}(0, \theta) = \frac{m^2 \lambda}{2}. \quad (2.8)$$

Up to first order in λ we find for the anomalous dimension of the corresponding YM operators⁵

$$\Delta - 4 = 2n \left[1 + \frac{1}{2} \frac{m^2 \lambda}{(2n)^2} \right]. \quad (2.9)$$

It is important to note that in this case the R -charge is zero. In order to take into account the R -charge, we consider pulsating string on S^5 which has a center of mass moving on the S^3 subspace of S^5 [11]. While in the previous

⁵See [10] for more details.

example the S^3 part of the metric we assumed trivial, now we consider all the S^3 angles to depend on τ (only). The corresponding Nambu-Goto action now is

$$S = -m\sqrt{\lambda} \int dt \sin \theta \sqrt{1 - \dot{\theta}^2 - \cos^2 \theta g_{ij} \dot{\phi}^i \dot{\phi}^j}, \quad (2.10)$$

where ϕ_i are S^3 angles and g_{ij} is the corresponding S^3 metric. The Hamiltonian in this case takes the form [11]

$$H = \sqrt{\Pi_\theta^2 + \frac{g^{ij}\Pi_i\Pi_j}{\cos^2 \theta} + m^2\lambda \sin^2 \theta}. \quad (2.11)$$

Once again, we see that the squared Hamiltonian looks like the one for a point particle, however, now the potential has angular dependence. Denoting the relevant quantum number of S^3 and S^5 by J and L correspondingly, one can write the Schrodinger equation for the free theory

$$-\frac{4}{\omega} \frac{d}{d\omega} \Psi(\omega) + \frac{J(J+1)}{\omega} \Psi(\omega) = L(L+4)\Psi(\omega), \quad (2.12)$$

where $\omega = \cos^2 \theta$. The solution to the Schrodinger equation is

$$\Psi(\omega) = \frac{\sqrt{2(l+1)}}{(l-j)!} \frac{1}{\omega} \left(\frac{d}{d\omega} \right)^{l-j} \omega^{l+j} (1-\omega)^{l-j}, \quad j = \frac{J}{2}; l = \frac{L}{2}. \quad (2.13)$$

The first order correction to the energy δE in this case is found to be

$$\delta E^2 = m^2 \lambda \frac{2(l+1)^2 - (j+1)^2 - j^2}{(2l+1)(2l+3)}, \quad (2.14)$$

or, up to first order in λ

$$E^2 = L(L+4) + m^2 \lambda \frac{L^2 - J^2}{2L^2} \quad (2.15)$$

The anomalous dimension then is given by

$$\gamma = \frac{m^2 \lambda}{4L} \alpha(2 - \alpha), \quad (2.16)$$

where $\alpha = 1 - J/L$.

We conclude this section referring for more details to [10] and [11].

3 Pulsating strings in warped S^4

In this section we investigate pulsating string solution in warped $AdS_6 \times S^4$ geometry. It is useful to look at the geometry from different points of view. From branes point of view this background can be obtained starting from N $D5$ branes wrapping a circle. After performing T -duality on the circle one ends up with $D4 - D8$ system. The location of the $D8$ branes gives the masses in the hypermultiplet while the region between two $D8$ branes is described by massive Type **II A** supergravity [20]. The later is also a solution to the low energy string equations. Ten dimensional background space is the warped product of AdS_6 and S^4 , i.e. it is a fibration of AdS_6 over S^4 and has the isometry group $SO(2, 5) \times SO(4)$.

To understand easily the relation to the five dimensional gauge theories, it is useful to make two steps consistent reduction. The first step is to integrate over the S^1 coordinate yielding $AdS_6 \times S^3$. Next, one can make reduction on S^3 gauging its isometry and ending up with AdS_6 . On other hand, Romans [21] constructed AdS_6 supergravity and the authors of [22, 23] made the connection with $D = 5$ superconformal field theory in the context of AdS/CFT. After these comments let us write the relevant geometry :

$$ds^2 = \frac{1}{2}W^2(\xi)[9(-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_4^2) + 4(d\xi^2 + \sin^2 \xi d\Omega_3^2)] \quad (3.1)$$

where

$$d\Omega_4^2 = d\theta_1^2 + \cos^2 \theta_1(d\theta_2^2 + \cos^2 \theta_2 d\phi_1^2 + \sin^2 \theta_2 d\phi_2^2) \quad (3.2)$$

$$d\Omega_3^2 = d\theta^2 + \cos^2 \theta d\psi_1^2 + \sin^2 \theta d\psi_2^2 \quad (3.3)$$

are the line elements on the unit S^4 and S^3 respectively. The non-trivial fields, i.e. RR four-form $F_{(4)}$ and the dilaton field supporting this solution, are:

$$F_{(4)} = \frac{20\sqrt{2}}{3}(\cos \xi)^{\frac{1}{3}} \sin^3 \xi d\xi \wedge \Omega_{(3)} \quad \text{and} \quad \exp \Phi = (\cos \xi)^{-\frac{5}{6}}. \quad (3.4)$$

The explicit form of the warp factor is given by

$$W(\xi) = (\cos \xi)^{-\frac{1}{6}} \quad (3.5)$$

The warp factor depends on the above mentioned S^1 coordinate ξ and we should note also that in string frame there are no singularities in the metric.

The AdS/CFT correspondence in this background beyond the supergravity approximation was further studied in [28]. The authors of [28] studied the rotating strings and pp-wave limit of this geometry. We proceed with the investigation of the class of pulsating strings in the warped background. The string configuration we will study is given by the following ansatz

$$t = \tau \quad \theta = m\sigma \quad \rho = \rho(\tau), \quad (3.6)$$

i.e. the string is spanned along θ direction and the time dependence of ρ realizes the pulsation of the string. Following the standard procedure developed in [10] we write the Nambu-Gotto action for the above metric (3.1) and string configuration (3.6)

$$S = -m\sqrt{\lambda} \int dt \frac{\sin \xi}{(\cos \xi)^{\frac{1}{3}}} \sqrt{9 \cosh^2 \rho - \dot{\rho}^2 - 4\dot{\xi}^2} \quad (3.7)$$

The canonical momenta that follow from this action are correspondingly

$$\Pi_\rho = \frac{m\sqrt{\lambda} \sin \xi}{(\cos \xi)^{\frac{1}{3}}} \frac{\dot{\rho}}{\sqrt{9 \cosh^2 \rho - \dot{\rho}^2 - 4\dot{\xi}^2}} \quad (3.8)$$

$$\Pi_\xi = \frac{m\sqrt{\lambda} \sin \xi}{(\cos \xi)^{\frac{1}{3}}} \frac{4\dot{\xi}}{\sqrt{9 \cosh^2 \rho - \dot{\rho}^2 - 4\dot{\xi}^2}} \quad (3.9)$$

After some calculations we get for the Hamiltonian the expression

$$H = 3 \cosh \rho \sqrt{\Pi_\rho^2 + \frac{\Pi_\xi^2}{4} + \frac{m^2 \lambda \sin^2 \xi}{(\cos \xi)^{2/3}}} \quad (3.10)$$

As in the previous section one can observe that H^2 looks like a point-particle Hamiltonian in which the last term serves as a potential. We will proceed as follows. First of all we will find the wave function for the free theory and the corresponding energy. After that we will quantize semiclassically the theory in order to obtain the corrections to the energy. We will consider the solutions around $\rho = 0$ but first of all we will find the eigenfunctions and eigenvalues of the wave equation corresponding to the warped $AdS_6 \times S^4$ geometry.

3.1 The wave function and corrections to the energy

Now we are going to derive the wave function corresponding to the free theory. The equation in ten dimensions that the wave function should satisfy is

$$\Delta_{10}F = \frac{1}{\sqrt{G}}\partial_\mu(\sqrt{G}G^{\mu\nu}\partial_\nu)F = 0 \quad (3.11)$$

where Δ_{10} is the Laplace-Beltrami operator for our metric and \sqrt{G} is

$$\sqrt{G} = \frac{9^3}{2}W^{10}(\xi)\sin^3\xi\cos\theta\sin\theta\cosh\rho\sinh^4\rho\cos^3\theta_1\cos\theta_2\sin\theta_2. \quad (3.12)$$

For further convenience let us write Δ_{10} in a more explicit way:

$$\Delta_{10}F = G^{00}\partial_0^2F + \frac{1}{\sqrt{G}}\partial_A(\sqrt{G}G^{AB}\partial_B)F + \frac{1}{\sqrt{G}}\partial_i(\sqrt{G}G^{ij}\partial_j)F \quad (3.13)$$

Here we have denoted the spatial part of AdS_6 indices by A, B , the indices on S^4 by i, j and the time by 0. We are considering motion only on the S^4 part of the geometry so the only non-trivial dependence will be on the coordinates of S^4 and on the time t . Dropping the trivial dependence on the transverse to S^4 coordinates, one can write the equation satisfied by the wave function as

$$G^{00}\partial_0^2F + \frac{1}{\sqrt{G}}\partial_i(\sqrt{G}G^{ij}\partial_j)F = 0 \quad (3.14)$$

Explicitly, the equation we have to solve for the free theory is

$$\partial_\xi^2F + \left(\frac{4}{3}\tan\xi + 3\cot\xi\right)\partial_\xi F + \frac{1}{\sin^2\xi}\Delta_3F + \frac{4E^2}{9}F = 0 \quad (3.15)$$

Here Δ_3 is the Laplace-Beltrami operator for the "usual" S^3

$$\Delta_3 = \frac{1}{\sin\theta\cos\theta}\partial_\theta(\sin\theta\cos\theta\partial_\theta) + \frac{1}{\cos^2\theta}\partial_{\psi_1}^2 + \frac{1}{\sin^2\theta}\partial_{\psi_2}^2 \quad (3.16)$$

with

$$\Delta_3\Psi(\theta) = -s(s+2)\Psi(\theta). \quad (3.17)$$

Now we can separate the variables in our equation (3.16) by making the ansatz $F(\xi, \theta) = \Phi(\xi)\Psi(\theta)$ and plug in the explicit form of the conformal factor and its derivative:

$$W(\xi) = (\cos\xi)^{-\frac{1}{6}} \quad \frac{dW}{d\xi} = \frac{1}{6}\sin\xi(\cos\xi)^{-\frac{7}{6}}. \quad (3.18)$$

The resulting equation is:

$$\frac{d^2\Phi(\xi)}{d\xi^2} + \left(\frac{4+5\cos^2\xi}{3\sin\xi\cos\xi}\right)\frac{d\Phi(\xi)}{d\xi} - \left(\frac{s(s+2)}{\sin^2\xi} - \frac{4}{9}E^2\right)\Phi(\xi) = 0 \quad (3.19)$$

where s is the angular momentum on S_3 and E is the energy. If we introduce a new variable $x = \tan^2\xi$ one can rewrite the equation (3.19) as

$$4x(1+x)\frac{d^2\Phi(x)}{dx^2} + \frac{2}{3}(12+13x)\frac{d\Phi(x)}{dx} - \left[\frac{s(s+2)}{x} - \frac{4E^2}{9(1+x)}\right]\Phi(x) = 0. \quad (3.20)$$

The solution to this equation is a linear combination of hypergeometric functions

$$\begin{aligned} \Phi(x) = C_1(1+x)^{\frac{5}{12}-\frac{\sqrt{25+16E^2}}{12}}x^{\frac{s}{2}} \times \\ {}_2F_1\left(\frac{s}{2}+\frac{5}{12}-\frac{\sqrt{25+16E^2}}{12}, \frac{s}{2}+\frac{19}{12}-\frac{\sqrt{25+16E^2}}{12}; s+2; -x\right) \\ + C_2(1+x)^{\frac{5}{12}-\frac{\sqrt{25+16E^2}}{12}}x^{-\frac{s}{2}-1} \times \\ {}_2F_1\left(\frac{7}{12}-\frac{s}{2}-\frac{\sqrt{25+16E^2}}{12}, -\frac{7}{12}-\frac{s}{2}-\frac{\sqrt{25+16E^2}}{12}; -s; -x\right) \end{aligned} \quad (3.21)$$

We should note that the second term in the solution should be dropped out because of $-s-2 < 0$ ⁶.

Therefore, if we introduce $J = \frac{\sqrt{25+16E^2}}{6} - \frac{19}{6}$ and $L = s$ one can rewrite $\Phi(\xi)$ as:

$$\Phi(\xi) = C_1x^{\frac{L}{2}}(1+x)^{-\frac{J}{2}-\frac{7}{6}}{}_2F_1\left(\frac{L-J}{2}, \frac{L-J-\frac{7}{3}}{2}; L+2; -x\right) \quad (3.22)$$

with square integrability conditions on $\Phi(\xi)$

$$J-L=k; \quad k=0, 1, 2 \dots \quad (3.23)$$

⁶One can compare (3.21) with the expression for the wave function in the case of "ordinary" S^4 :

$$\Psi_L^J(\xi) = \tan^L\xi \cos^J\xi {}_2F_1\left[\frac{L-J}{2}, \frac{L-J+1}{2}; L+2; -\tan^2\xi\right]$$

where L and J are the angular momenta on S^3 and S^4 respectively.

The integrability conditions (3.23) mean that the wave function can be expressed in terms of Jacobi polynomials by making use of the relation:

$$P_n^{(\alpha,\beta)}(x) = \frac{(n+\alpha)!}{(\alpha)!(n)!} \left(\frac{1+x}{2}\right)^n {}_2F_1(-n, -n-\beta; \alpha+1; \frac{x-1}{x+1}) \quad (3.24)$$

It is obvious that in our case $n = \frac{J-L}{2} = \frac{k}{2}$, $\alpha = L+1$, $\beta = \frac{7}{6}$. It is useful also to change the variable in our solution, by introducing:

$$x = \frac{1-y}{1+y} \rightarrow 1+x = \frac{2}{1+y} \quad (3.25)$$

and rewrite Φ in the new variable as:

$$\Phi(y) = C_1 \left(\frac{1-y}{2}\right)^{\frac{\alpha-1}{2}} \left(\frac{1+y}{2}\right)^{\frac{7}{6}} \left(\frac{1+y}{2}\right)^n {}_2F_1[-n, -n-\beta; \alpha+1; \frac{y-1}{y+1}], \quad (3.26)$$

or:

$$\Phi(y)_n = \frac{C_1(n)!(\alpha)!}{(n+\alpha)!} \left(\frac{1-y}{2}\right)^{\frac{\alpha-1}{2}} \left(\frac{1+y}{2}\right)^{\beta} P_n^{(\alpha,\beta)}(y) \quad (3.27)$$

The normalized wave functions $\Phi_n(y)$ should satisfy orthogonality conditions

$$\int \frac{1}{W^2(y)} \Phi_n(y) \Phi_m(y) d\mu(y) = \delta_{nm}. \quad (3.28)$$

In (3.28) $W^2(y)$ is the conformal factor and $d\mu(y)$ is the invariant volume (we have integrated out all the dependence on other coordinates except $y = \cos^2 \xi - \sin^2 \xi$). This leads to the explicit form of the orthogonality condition in the new variable:

$$\frac{C_1^2 9^3 (n)!(m)!(\alpha)!^2}{2^{\alpha+\beta+\frac{11}{6}} (n+\alpha)!(m+\alpha)!} \int (1-y)^\alpha (1+y)^\beta P_n^{(\alpha,\beta)}(y) P_m^{(\alpha,\beta)}(y) dy = \delta_{nm} \quad (3.29)$$

By using the standard normalization condition for the Jacobi polynomials:

$$\int (1-y)^\alpha (1+y)^\beta [P_n^{(\alpha,\beta)}(y)]^2 dy = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(n)!\Gamma(n+\alpha+\beta+1)} \quad (3.30)$$

one can determine C_1 :

$$C_1^2 = \frac{2^{\frac{5}{6}}}{9^3} (2n+\alpha+\beta+1) \frac{\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\beta+1)\Gamma(\alpha+1)} \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} \quad (3.31)$$

Using that $J = \frac{\sqrt{25+16E^2}}{6} - \frac{19}{6}$, the expression for the energy reads off

$$E^2 = (J + \frac{7}{3})(J + 4). \quad (3.32)$$

We can now find the corrections to this energy by using the semiclassical approximation:

$$\delta E^2 = \frac{C_1^2 9^3 (n!)^2 (\alpha!)^2}{2^{\alpha+\beta+\frac{11}{6}} [(n+\alpha)!]^2} m^2 \lambda \int_{-1}^1 (1-y)^{\alpha+1} (1+y)^{\beta-1/3} [P_n^{(\alpha,\beta)}(y)]^2 dy \quad (3.33)$$

Let us first calculate the integral:

$$I = \int_{-1}^1 (1-y)^{\alpha+1} (1+y)^{\beta-1/3} [P_n^{(\alpha,\beta)}(y)]^2 dy \quad (3.34)$$

It is useful to change variables by introducing $y = 1 - \frac{x^2}{2n^2}$ after which our integral becomes:

$$I = \int_0^{2n} \frac{x^{2\alpha+3}}{2^{\alpha+1} n^4} (2 - \frac{x^2}{2n^2})^{\beta-1/3} [\frac{1}{n^\alpha} P_n^{(\alpha,\beta)}(1 - \frac{x^2}{2n^2})]^2 dx \quad (3.35)$$

Using the well known asymptotic behavior of the Jacobi polynomials for large n we get:

$$[\frac{1}{n^\alpha} P_n^{(\alpha,\beta)}(1 - \frac{x^2}{2n^2})] \approx (\frac{2}{x})^\alpha J_\alpha(x) \quad (3.36)$$

where J_α are the Bessel functions of first kind. The substitution of (3.36) into the integral (3.35) gives:

$$I = \int_0^{2n} \frac{x^3 2^{\alpha-1}}{n^4} (2 - \frac{x^2}{2n^2})^{\beta-1/3} [J_\alpha(x)]^2 dx \quad (3.37)$$

The explicit solution is in terms of degenerate hypergeometric function:

$$I = 2^{\alpha+\beta+5/3} (\alpha+1) n^{2\alpha} \frac{\Gamma(\beta+2/3)}{\Gamma(\alpha+\beta+8/3) \Gamma(\alpha+1)} \times {}_2F_3[\alpha+2, \alpha+1/2; \alpha+1, 2\alpha+1, \alpha+\beta+8/3; -4n^2] \quad (3.38)$$

For large α one can use the approximation:

$$\begin{aligned} {}_2F_3[\alpha+2, \alpha+\frac{1}{2}; \alpha+1, 2\alpha+1, \alpha+\beta+\frac{8}{3}; -4n^2] &\rightarrow \\ &\rightarrow {}_2F_1[\alpha+2, \alpha+\frac{1}{2}; \alpha+1; -\frac{-4n^2}{(2\alpha+1)(\alpha+\beta+\frac{8}{3})}] \end{aligned} \quad (3.39)$$

To evaluate the correction to the energy (large n) we use the transformation of the hypergeometric function:

$$\begin{aligned} {}_2F_1[a, b; c; z] &= \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a} {}_2F_1[a, 1-c+a; 1-b+a; \frac{1}{z}] + \\ &\quad \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b} {}_2F_1[b, 1-c+b; 1-a+b; \frac{1}{z}]. \end{aligned} \quad (3.40)$$

Evaluating the leading order behavior of the correction to the energy, we find:

$$\delta E^2 \approx \left(\frac{m^2 \lambda}{2n} + (\alpha/2 + 1/4)(2\alpha + 1)(\alpha + \beta + 8/3) \frac{m^2 \lambda}{4n^3} \right). \quad (3.41)$$

3.2 Bohr-Sommerfeld quantization

For completeness (and consistency check) we will give here also the Bohr-Sommerfeld quantization procedure applied to our case. First of all we point out that our potential is obviously even, so the quantized states satisfy:

$$(2n + \frac{1}{2})\pi = \int_{-\xi_0}^{\xi_0} \sqrt{E^2 - m^2 \lambda \frac{\sin^2 \xi}{(\cos \xi)^{2/3}}} d\xi \quad (3.42)$$

In (3.42) the values $\pm \xi_0$ are the turning points of the potential, which are the solutions to the equation:

$$E = m\sqrt{\lambda} \frac{\sin \xi_0}{(\cos \xi_0)^{1/3}} \quad (3.43)$$

For large values of the energy E we find that $\xi_0 = \pi/2$. It is appropriate to use the notation $B = E/m\sqrt{\lambda}$, so if we now define:

$$y = B^{-1}(\tan \xi)^{1/3} \quad (3.44)$$

one can get for the measure $d\xi$:

$$d\xi = 3B(\tan \xi)^{2/3} \cos^2 \xi dy. \quad (3.45)$$

With this at hand, we transform our integral to:

$$(2n + \frac{1}{2})\pi = 3B^3 m\sqrt{\lambda} \int_0^{y_0} \sqrt{\left(B^2 - \frac{y^6 B^6}{(1+y^6 B^6)^{2/3}}\right) \frac{y^4}{(1+y^6 B^6)^2}} dy. \quad (3.46)$$

After some transformations and using the fact that $B \rightarrow \infty$ we reduce our integral to:

$$(2n + \frac{1}{2})\pi = \frac{3}{B^2}m\sqrt{\lambda} \int_0^1 \frac{y^2\sqrt{1-y^2}}{y^6 + \frac{4}{3B^6}} dy. \quad (3.47)$$

This integral is exactly solvable in terms of degenerate hypergeometric functions ${}_3F_2(-\frac{4}{3B^6})$. Using the series expansion for this function for small argument and taking into account only the terms of order $\mathcal{O}(1/E)$ we end up with:

$$\frac{\sqrt{3}}{4}E - \frac{3^{1/6}4^{1/3}}{4} \frac{m^2\lambda}{E} = 2n + \frac{1}{2}. \quad (3.48)$$

Solving this equation for E and using that E is very large we find that the energy is:

$$E = \frac{4n+1}{\sqrt{3}} + \frac{3^{1/6}4^{1/3}}{2} \frac{m^2\lambda}{2n} \quad (3.49)$$

So the classical energy, which through the correspondence is identified with the bare dimension of the corresponding gauge operators, is proportional to n :

$$\Delta_b = \frac{4n+1}{\sqrt{3}} \quad (3.50)$$

According to the AdS/CFT correspondence, the anomalous dimension of the corresponding SYM operator includes also the corrections to the energy. In our case we find correspondingly that:

$$\Delta - \Delta_b \approx \frac{m^2\lambda}{2n} \quad (3.51)$$

As expected, the result obtained by making use of Bohr-Sommerfeld quantization is the same as the one calculated by using perturbation theory. We can compare our solution to that of Minahan [10] for pulsating string on pure S^5 . We see that the bare dimension has the same behavior but the numerical coefficient is different which is expected because our sphere S^4 is deformed by the conformal factor. The anomalous dimension also has the same behavior but with different numerical coefficient.

4 Conclusions

In this section we summarize the results of our study. The goal we pursued in this paper was to investigate the pulsating strings in type IIA background

described in section 3. The motivation was the significant difference between the results found for $AdS_5 \times S^5$ background and those obtained in [28] for the case of warped $AdS_6 \times S^4$. We looked for the simplest pulsating string solutions in warped S^4 part of $AdS_6 \times S^4$ background. Using the simple string ansatz

$$t = \tau \quad \theta = m\sigma \quad \rho = \rho(\tau), \quad (4.1)$$

we find corresponding pulsating string solutions. Following the procedure developed in [10], we find the energy. To obtain the energy corrections we used the general approach suggested by Minahan [10]. For this purpose we consider the Nambu-Goto actions and find the Hamiltonian. After that we quantize the resulting theory semiclassically and obtain the corrections to the energy. We calculate the corrections to the energy by making use of two approaches, namely perturbation theory and Bohr-Sommerfeld quantization. Both approaches give the same result

$$\delta E^2 \approx \left[\frac{m^2 \lambda}{2n} + (\alpha/2 + 1/4)(2\alpha + 1)(\alpha + \beta + 8/3) \frac{m^2 \lambda}{4n^3} \right]. \quad (4.2)$$

One can compare our solution to that of Minahan [10] for pulsating string on pure S^5 . The comparison gives that the bare dimension has the same behavior but the numerical coefficients are different. This is expected since our sphere S^4 is deformed by the conformal factor $W(\xi)$. From AdS/CFT point of view the corrections to the classical energy enter the anomalous dimensions of the operators in SYM theory and therefore they are of primary interest. Therefore the anomalous dimensions also have qualitatively the same behavior as in $AdS_5 \times S^5$ but with different numerical coefficients.

As a final comment we note that to complete the analysis from AdS/CFT point of view, it is of great interest to develop an analysis allowing to compare our result to that in SYM side. We leave this important question for future research.

Acknowledgements: R.R. would like to thank C. Nunez for comments and suggestions.

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